Natural multiparticle entanglement in a Fermi gas

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We investigate multipartite entanglement in a non-interacting fermion gas, as a function of fermion separation, starting from the many particle fermion density matrix. We prove that all multiparticle entanglement can be built only out of two-fermion entanglement. Although from the Pauli exclusion principle we would always expect entanglement to decrease with fermion distance, we surprisingly find the opposite effect for certain fermion configurations. The von Neumann entropy is found to be proportional to the volume for a large number of particles even when they are arbitrarily close to each other. We will illustrate our results using different configurations of two, three, and four fermions at zero temperature although all our results can be applied to any temperature and any number of particles.

Introduction. Quantum entanglement plays a crucial role in quantum mechanics, and is extensively used in quantum information. However, it was only recently that researchers have started to investigate entanglement in systems containing a large number of particles. This is of fundamental importance because entanglement was found to be relevant not only in microscopic systems, but also on a macroscopic scale [1, 2]. Multipartite entanglement seems to play an important role in condensed matter systems, and might be the key ingredient to the solution of unresolved physical problems such as high temperature superconductivity [3]. In this work we investigate multipartite entanglement in a non-interacting Fermi gas. Bipartite entanglement in this simple quantum system has already been shown to be fully characterized by the exchange integral due to the antisymmetry of the wavefunction [4, 5, 6].

In this letter we will show that all multiparticle entanglement can be built only from bipartite entanglement. This is a significant result because it shows that a complete description of quantum correlations at all levels is possible in a realistic many body system such as a non-interacting Fermi gas. We show that the n-particle density matrix can be written as a sum of the completely mixed state and a mixture of all possible two fermion antisymmetrized wavefunctions. We use entanglement witnesses to illustrate that genuine tripartite entanglement does not exist in this system, in agreement with our previous expansion of the density matrix. We then investigate bipartite entanglement for three and four fermions for different fermion configurations. This entanglement is quantified using the negativity [7]. Finally we show that for large number of fermions the entropy is always proportional to this number (which in turn is proportional to the volume of the system), independently of the fermion distance. For a small number of particles and small fermion separation the entropy is smaller than this number (volume). This clearly establishes the fact that entanglement of a non-interacting Fermi gas can be treated like any other macroscopic physical quantity and that it can be related to other macroscopic observables such as the volume of the gas or number density. Any mean field theory ignoring entanglement when describing macroscopic effects in many body systems is therefore unlikely to be successful even when, remarkably, the constituents of the system are non-interacting as in our case.

Density matrix. We consider a many fermion system with a fixed number of particles and a density matrix ρ. The elements of the reduced density matrices for $1, 2, 3...n$ particles labeled by ρ_1, ρ_2, $\rho_3...\rho_n$ respectively are given by [8]:

$$\langle 1|\rho_1|1'\rangle = \langle\Psi^\dagger(1')\Psi(1)\rangle$$
$$\langle 12|\rho_2|1'2'\rangle = \langle\Psi^\dagger(2')\Psi^\dagger(1')\Psi(1)\Psi(2)\rangle$$
$$\langle 123|\rho_3|1'2'3'\rangle = \langle\Psi^\dagger(3')\Psi^\dagger(2')\Psi^\dagger(1')\Psi(1)\Psi(2)\Psi(3)\rangle$$
$$\langle 1...n|\rho_n|1'...n'\rangle = \langle\Psi^\dagger(n')\Psi^\dagger((n-1)')...\Psi(n-1)\Psi(n)\rangle$$

where $1 \equiv (\mathbf{r}_1, \sigma_1)$, $\mathbf{r}_1$ is the position vector and $\sigma_1 = \uparrow, \downarrow$ is the spin of the fermion. The average is given by $\langle...\rangle = Tr\{\rho...\}$. For the sake of simplicity all our results are illustrated at zero temperature, where $\rho = |\phi_0\rangle\langle\phi_0|$, with $|\phi_0\rangle = \prod_k^{k_F} c_{k,\sigma}^\dagger|vac\rangle$ equals the ground state of the Fermi system. The $c_{k,\sigma}^\dagger$ is the creation operator that creates an electron of momentum k and spin σ. The Fermi momentum is denoted by k_F and the vacuum state is $|vac\rangle$. The Ψ are the field operators and obey the usual fermion anti-commutation relations $\{\Psi_{\sigma_1'}^\dagger(\mathbf{r}_1'), \Psi_{\sigma_1}(\mathbf{r}_1)\} = \delta_{\sigma_1',\sigma_1}\delta(\mathbf{r}_1 - \mathbf{r}_1')$.

After a somewhat lengthy but straightforward calculation we arrive at a form for the density matrix for n particles which is particulary useful to investigate entanglement:

$$\rho_n = (1 - \sum_{ij} p_{ij})\frac{\mathbf{I}}{2^n} + \sum_{ij} p_{ij}|\Psi_{ij}^-\rangle\langle\Psi_{ij}^-| \otimes \frac{\mathbf{I}}{2^{n-2}} \quad (1)$$

where $|\Psi_{ij}^-\rangle = 1/\sqrt{2}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ is the maximally entangled singlet state of the pair ij. The sum runs over all the pairs ij. The probabilities p_{ij} are functions of the relative distances between all pairs. As an example, we write down the density matrix for the two and three particle case:

$$
\begin{aligned}
\rho_2 &= p\frac{\mathbf{I}}{4} + (1-p)|\Psi^-\rangle\langle\Psi^-| \\
\rho_3 &= (1 - p_{12} - p_{13} - p_{23})\frac{\mathbf{I}}{8} + p_{12}|\Psi_{12}^-\rangle\langle\Psi_{12}^-| \otimes \frac{\mathbf{I}}{2} \\
&+ p_{13}|\Psi_{13}^-\rangle\langle\Psi_{13}^-| \otimes \frac{\mathbf{I}}{2} + p_{23}|\Psi_{23}^-\rangle\langle\Psi_{23}^-| \otimes \frac{\mathbf{I}}{2}
\end{aligned}
\tag{2}
$$

where $p = (2 - 2f(r)^2)/(2 - f(r)^2)$ and $f(r) = j_1(x)/x$, with the Bessel function $j_1(x) = (\sin x - x\cos x)/x^2$ and $x = k_F r$. The relative distance between the fermion pair is denoted by r. The function $f(r)$ is one for $r = 0$ and zero for large r. For three fermions, we have three different pairs and for the pair ij : $p_{ij} = (-f_{ij}^2 + f_{ij}f_{ik}f_{jk})/(-2 + f_{ij}^2 + f_{ik}^2 + f_{jk}^2 - f_{ij}f_{ik}f_{jk})$. The function f_{ij} is a function of the relative distance between fermion i and j only. Note that the probabilities p_{ij} can be calculated for any number of particles.

Entanglement. The Peres-Horodecki criterion is the condition for the existence of entanglement in the two particle case [9]. In our earlier work, we found this to imply that $f(r)^2 > \frac{1}{2}$. This means that two electrons are entangled if the relative distance between them is smaller then $1.8/k_F$ for $T = 0$ [6]. Two fermions are maximally entangled if they are at the same position. This is because of the Pauli exclusion principle. In general, since the overall state must be antisymmetric, if the spatial wavefunctions fully overlap and thus are symmetric, then the spins must be antisymmetrised. The fermions must, therefore, be in the maximally entangled spin singlet state $|\Psi^-\rangle$. We will show that such two-particle entanglement is also the main building block for multi-particle entanglement. In order to illustrate this behavior we will now consider entanglement in systems containing three fermions .

Tripartite entanglement. From the decomposition of the density matrix it is clear that no genuine tripartite entanglement exists. We now formally show this using the method of entanglement witnesses. These are observables which (by our convention) have a positive expectation value for all separable states, and a negative expectation value for some entangled states, i.e entanglement exists if $Tr\{\rho^{(3)}\Pi\} < 0$, where Π is the witness [10]. It has been shown that there are only two different classes of tripartite entanglement, which are represented by the $|GHZ\rangle = 1/\sqrt{2}(|\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\rangle)$ and $|W_3\rangle = 1/\sqrt{3}(|\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle + |\uparrow\uparrow\downarrow\rangle)$ states [10]. The corresponding witnesses are defined as: $\Pi_{GHZ} = 1/2 - |GHZ\rangle\langle GHZ|$ and $\Pi_{W_3} = 2/3 - |W_3\rangle\langle W_3|$. For both of these witnesses,

because $0 < |f_{ij}|^2 < 1$, the trace of $\rho^{(3)}\Pi$ cannot be negative. This confirms that genuine tripartite entanglement does not exist in the ideal Fermi gas.

Bipartite entanglement . We will now investigate if there is entanglement between two groups of fermions. One group contains fermion i and the other group the fermion pair jk. As a measure of entanglement we use the negativity, defined by $N_{[i,jk]} = (\|\rho_3^{T_i}\|_1 - 1)/2$, where $\|\rho_3^{T_i}\|_1$ is the trace norm of the partial transpose of the reduced density matrix ρ_3 of fermion i versus the other two jk and denote it as $N_{[i,jk]}$. The trace norm can be evaluated to be $\|\rho_3^{T_i}\|_1 = 1 + 2|\sum_l \mu_l|$, where the sum goes over the negative eigenvalues of the partial transpose. There are eight eigenvalues in total and only two of them are negative having the same value λ. The negativity is, therefore, $N_{[i,jk]} = 2|\lambda|$. Negativity is a good measure of bipartite entanglement because it is monotonic under local operations and classical communication and is also equal to zero if fermion i is not entangled to the fermion pair jk. It reaches its maximal value of $1/2$ if fermion i is maximally entangled to the fermion pair jk. We will now investigate different arrangements of three fermions, and investigate the behavior of negativity.

We first consider three fermions on a straight line. The distance between the fermion i and the fermion j is fixed. The remaining fermion k moves from the position of fermion i, to the position of fermion j, i.e from $x = 0$ to $x_{max} = k_F r_{ij}$, where r_{ij} is the relative position between fermion i and fermion j. At $x = 0$, fermion k is maximally entangled to fermion i, and can therefore not be entangled to fermion j, independent of r_{ij}. The state of the total system is then $\rho_3 = |\Psi_{ik}^-\rangle\langle\Psi_{ik}^-| \otimes \frac{\mathbf{I}}{2}$. As fermion k moves away from fermion i we expect the negativity $N_{[k,ij]}$ to first drop but and then to increase again as fermion k approaches j. This is confirmed in Fig.1.

We next consider the case where the fermions are located on the edges of an isosceles triangle. Fermions i and j form the base of the triangle which is fixed. Fermion k is moved away from the midpoint of the base. The entanglement negativity $N_{[k,ij]}$ and $N_{[i,jk]}$ for this scenario are plotted in Fig.2. The entanglement $N_{[k,ij]}$ monotonically decreases as k moves away from ij because the effect of antisymmetrization becomes weaker with the distance (dashed line in Fig. 2). The entanglement negativity $N_{[i,jk]}$ (solid line in Fig.2) initially follows the same trend as $N_{[k,ij]}$ for exactly the same reason. Surprisingly however, the entanglement $N_{[i,jk]}$, after reaching its minimum value, starts to increase and then reaches its saturation value. The reason for this is the following. When fermion k is further away from i and j than the distance between fermion i and j itself, the effect of antisymmetrization between i and j on entanglement is larger than the effect of antisymmetrization between i and k. If then the distance is further increased, the position of fermion k has a vanishingly small role on entangle-
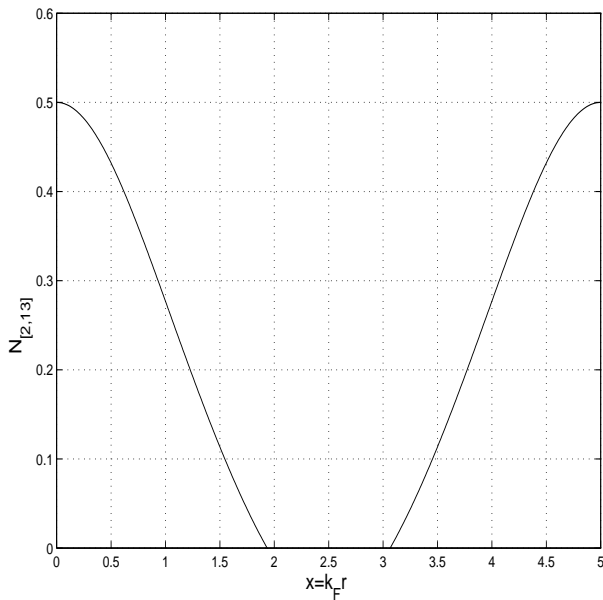
FIG. 1: The negativity $N_{[2,13]}$ is plotted for fermion 2 moving from fermion 1 to fermion 3. This corresponds to the text values $i = 1$, $j = 3$, $k = 2$. The distance between 1 and 3 is fixed to $x_{max} = 5$. This is a dimensionless number.
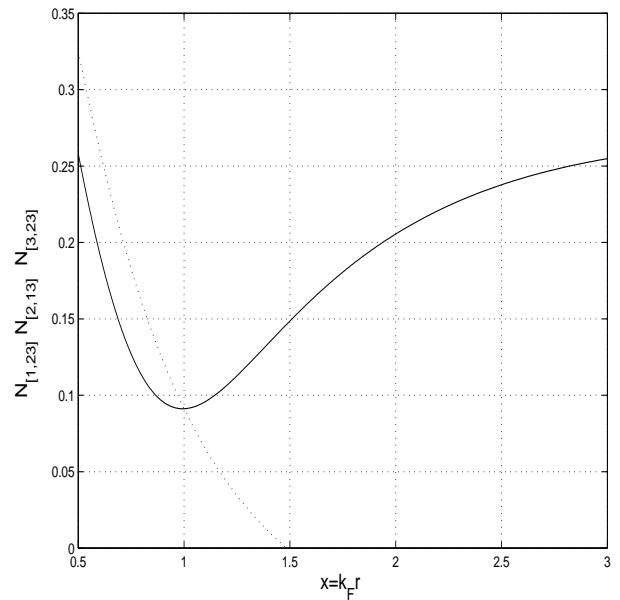

FIG. 2: The negativities $N_{[2,13]} = N_{[3,12]}$ (solid), $N_{[1,23]}$ for an isosceles triangle, for which fermions 2 and 3 form the base are plotted. Fermion 1 is moving away from the midpoint of the base. This corresponds to the text values $i = 2$, $j = 3$ and $k = 1$.

ment. The three particle density matrix then becomes: $\rho_3 \sim \rho_2 \otimes \mathbf{I}/2$, where ρ_2 is the reduced density matrix of the pair ij. Note that the minimum of negativity is reached when the fermions are equally distant from each other. This is because entanglement is monogamous and each particle has to share entanglement equally with the other two. This case will now be analyzed in more detail.

We now consider the case when the fermions are separated by equal distances. For three particles the fermions are located on the edges of an equilateral triangle, and for the four particle case on the edges of an tetrahedron. Entanglement in this case is plotted Fig.3. We start by putting the fermions in a very small volume of radius ε. Because of the Pauli exclusion principle, only two fermions can be in the same location. More then two fermions would mean that at least two quantum numbers are the same, which is forbidden. As the distance between the fermions increases, $N_{[i,jkl]} < N_{[l,mn]} < N_{[p,q]}$ at all distances, because the entanglement is shared between the fermion pairs, and the more fermions are involved the less entanglement we gain. All of this can be generalized to an arbitrary number of fermions. If the fermions are all in a small volume of radius ε, then the state is in an equal mixture of all the singlet states of the pairs, and only the second term of (1) survives. Higher order entanglement does exist, but the Pauli exclusion principle forbids maximally entangled states other then the $|\Psi^-\rangle$. This is the reason why we do not have $|GHZ\rangle$ or $|W_3\rangle$ states in the system. If the fermions are fur-

ther away from each other the first term in (1) becomes important and the total state ρ_n becomes even less entangled. The interplay between the two terms in the density matrix is also important when we want to calculate the total entropy of the fermions $S(n) = -Tr\{\rho_n \ln \rho_n\}$. We study this quantity, because at $T = 0$ it quantifies the amount of entanglement between the measured electrons and the remaining unmeasured electrons.

Entropy. We now investigate the von Neumann entropy of the Fermi system as a function of fermion distance as shown in Fig.4. At $T = 0$, the system is in a pure state. For the two particle case, the entropy is zero for $x = 0$ because the two fermions are in the pure singlet state. It then increases to two as the distance increases. This behavior is also observed for three particles as well as four. In these cases it does not start at zero because we cannot have more then two fermions in the same location, but again reaches the value of three (four) as the distance increases. For large n, the entropy becomes nearly equal to the number of particles even for very small distances. For a large particle system at low constant density, the reduced density matrix of the system is then given by $\rho_n = \mathbf{I}/2^n$ and the entropy is $S(n) = n \ln 2$ which is proportional to the volume of the system. For a dense system the density matrix is given by: $\rho_n = \sum_{ij} |\Psi^-_{ij}\rangle\langle\Psi^-_{ij}| \otimes \mathbf{I}/2^{n-2}$. The entropy of this state is also proportional to the number of fermions for large n. Only if the number of fermions is small and if they are very close, the Pauli exclusion principle pre-
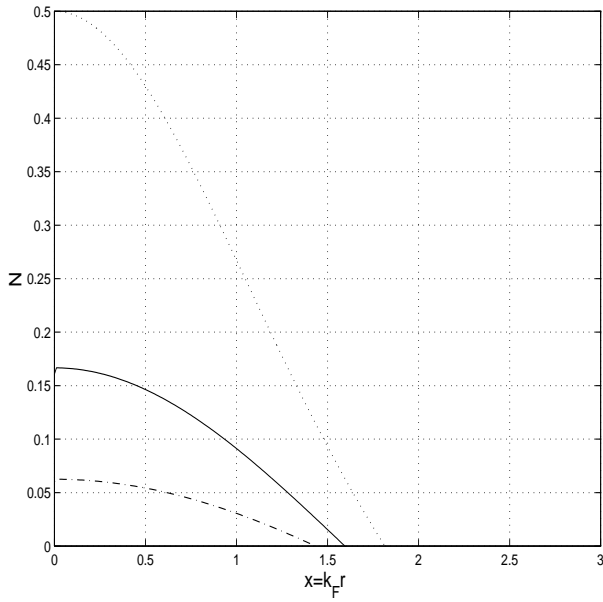
FIG. 3: The negativities $N_{[1,2]}$, $N_{[1,23]}$(solid) and $N_{[1,234]}$(dashed) are plotted. This corresponds to text values $p, q = 1, 2$, $l, mn = 1, 23$, and $i, jkl = 1, 234$.
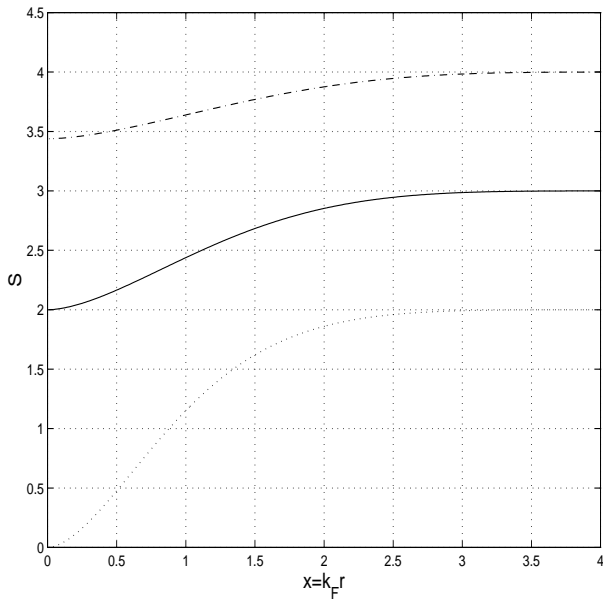

FIG. 4: The von-Neumann entropy S_2(dashed), S_3(solid), S_4 is plotted as a function of fermion distance. This behavior is analytically explained in the text.

vents the entropy from being proportional to the number of fermions. This can be explained as follows: If all the states are equally likely the entropy is the logarithm of the number of all possible configurations of these states. Since we have a system of fermions they have to be an-

tisymmetrized and therefore this number is equal to the total number of states 2^n, minus the number of symmetric states $(n + 1)$. The entropy therefore is:

$$S(n) = \ln(2^n - (n+1)) \qquad (3)$$

We can see a complete agreement between this formula and the entropy in Fig. 4 for $x = 0$. It is also clear that if the number of fermions is large, then the first term in the entropy dominates, giving us the previous result of entropy being proportional to the volume of the system.

Conclusion. We have presented a form of the density matrix for n fermions in an ideal fermi gas which is particulary useful to investigate entanglement in this system. We then showed that no genuine multipartite entanglement exists, and that all multipartite entanglement can be built only from the bipartite entanglement between fermion pairs. Lastly, we showed that the entropy of a large Fermi gas is always proportional to its volume, independently of fermion distance. It is only for a small number of fermions and small distances that the entropy is smaller than this number (volume). We believe that our work shows that multipartite entanglement in complex macroscopic systems can be studied and even fully understood with the existing techniques of quantum information. We hope that this stimulates other studies in similar directions of solid state and condensed matter systems.

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